Digital Communications Engineering 1

(COMM2108)

Baseband Communication

Digital Receivers

Digital Receivers

- Having dealt with the formatting of analogue waveforms into digital data and then the conversion of digital data into electrical waveforms (or *symbols*), the next step is the detection of the symbols and the recovery of the digital data from the received waveform.
- However, due to the inevitable presence of noise in the system, errors may occur in the detection and recovery of the digital data.

Digital Receivers

- *Noise* refers to unwanted electrical signals that are always present in electrical systems.
- Presence of noise superimposed on a signal tends to obscure or mask the signal which limits the receiver's ability to make correct symbol decisions.
- Noise arises from a variety of sources, both man-made and natural.
- *Man-made* noise includes switching transients and other radiating EM signals.
- *Natural* noise includes electrical circuit and component noise, atmospheric disturbances and galactic sources.
- Good engineering design can eliminate much of the noise or its undesirable effect through *filtering* and *shielding*.

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- However, there is one natural source of noise, called *thermal* or *Johnson noise*, that cannot be eliminated.
- Thermal noise is caused by the random thermal motion of electrons in all dissipative components, e.g. resistors, wires etc.
- We can describe thermal noise as a zero-mean *Gaussian* random process.

Digital Receivers

- In a digital transmission system, in any given signaling interval, one of M possible waveforms are transmitted.

$$s_i(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\} \text{ for } 0 \leq t \leq T$$

- At the receiver, a signal $r(t)$ is received and is given by:

$$r(t) = s_i(t) + n(t)$$

- where $n(t)$ is Additive White Gaussian Noise (AWGN) which causes the original transmitted signal $s_i(t)$ to become corrupted during its propagation over the communications channel.

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- *Additive* => the noise “adds” onto the signal.
- *White* => the noise has a flat or “white” spectrum that has a power spectral density of $N_0/2$ (W/Hz) that extends from -∞ Hz to +∞ Hz.
- *Gaussian* => the amplitudes of the noise voltage fluctuations follows a Gaussian distribution.

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- Since noise is essentially a random process (i.e. like tossing a fair coin), it can only be properly described in terms of its statistical properties.
- These include measures like the mean (or average) value, the variance (or standard deviation) or its *probability density function* (pdf).
- The pdf describes the probability that the noise voltage will have a particular value.
- For example, when tossing a fair coin the probability of it landing heads or tails is equal to ½ , i.e.

$$p(H) = p(T) = 0.5$$

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- Since the noise $n(t)$ is assumed to be AWGN, its probability density function is given by

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2}$$

- The noise $n(t)$ has zero mean and a variance (or average noise power) of σ^2.

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- The expression for the probability density function $p(n)$ shows that the noise voltage amplitudes are distributed according to a Gaussian distribution.
- From the plot of $p(n)$, it can be seen that the most probable amplitudes are those with small positive or negative values.
- Furthermore, in theory the noise amplitude can be infinitely large, but very large noise amplitudes will have a very small probability.

Digital Receivers

- *Deterministic* signals where there is no uncertainty with respect to its values at any time, i.e. its value is predictable or can be calculated in advance. Deterministic signals can be modeled by explicit mathematical expressions such as

$$x(t) = 10\cos(2t)$$

- *Random* signals where there is some uncertainty in terms of its value before the signal occurs. Random signals cannot be described by such an explicit mathematical expression. Instead, one can only meaningfully describe the signal in terms of its statistical properties such as the mean, variance, or probability density function (*pdf*).

Digital Receivers

- Since $n(t)$ is a randomly varying quantity, the received signal $r(t)$ will also be a randomly varying quantity.
- Consequently, there will be a degree of uncertainty as to the value of $r(t)$ which will lead to errors in the recovery of the digital data.
- The function of the digital receiver is to determine *which* of the signaling waveforms (or symbols) was transmitted in any given signaling interval.

Digital Receivers

- In general, the function of a digital receiver is to determine *which* signal was transmitted (and not what signal).

- Analogue receivers are used to determine *what* signal was transmitted.

- This represents a fundamental difference in the operation of digital receivers and analogue receivers.

- In a digital receiver, the set of transmitted signals, i.e. the set of waveforms $\{s_1(t), s_2(t), \ldots, s_M(t)\}$, is known *a priori* (or beforehand) to the receiver.

Digital Receivers

- In other words, the digital receiver knows what to expect, however it does not know which signal is being transmitted in any given signaling interval.

- Therefore, the function of the digital receiver is to determine *which* signal from the set of M possible signals is being transmitted.

- It is this feature of digital receivers that gives them a superior noise performance over analogue receivers.

Digital Receivers

- There are two basic steps in the detection of digital signals:
- ***Step 1*:** The reduction of the received waveform $r(t)$ to a fixed quantity $z(T)$ known as a *test statistic*. This operation is performed by a *linear filter* followed by a *sampler*.

Digital Receivers

- ***Step 2*:** The test statistic $z(T)$ is compared with a reference or threshold value γ to determine which signal was transmitted.

$$z(T) \underset{s_2}{\overset{s_1}{\gtrless}} \gamma$$

- If $z(T) > \gamma$, signal $s_1(t)$ is considered to have been transmitted.
- If $z(T) < \gamma$, signal $s_2(t)$ is considered to have been transmitted.

Digital Receivers

- Occasionally, due to the effects of noise, it can sometimes happen that:

  $z(T) < \gamma$ when $s_1(t)$ has been transmitted which will result in the receiver making an incorrect decision and an error occurs.

- Similarly, it can sometimes also happen that:

  $z(T) > \gamma$ when $s_2(t)$ has been transmitted which will result in the receiver making an incorrect decision and an error occurs.

Digital Receivers

- Since the noise $n(t)$ is a random variable, the received signal $r(t)$ will also be a random variable.
- Consequently, the linear filter output $z(t)$ and $z(T)$, its sampled value at $t = T$, will also be a random variable.
- Since $z(T)$ is a random variable, it can only be meaningfully considered in terms of its statistics (i.e. its mean or variance) or in terms of its probability density function.

Digital Receivers

- The analysis of digital receivers begins with the important concept of the *matched filter*.
- The matched filter is a linear time-invariant filter that will be shown to lead to the optimum detection of a signal waveform that is immersed in additive white Gaussian noise (AWGN).
- Here *optimum* refers to the minimum probability of an error occurring.
- The term “matched filter” refers to the fact the impulse response of the filter is “matched” to the signal waveform – the filter is designed to specifically detect the presence of the signal $s_i(t)$ that is buried in the noisy received signal $r(t)$.

Digital Receivers

- More precisely, the matched filter is designed to maximise the signal-to-noise ratio (*SNR*) at the filter output for a given waveform $s_i(t)$ at the sampling instant $t = T$.
- The filter input is the received signal $r(t)$ which consists of the transmitted signal $s_i(t)$ corrupted by AWGN $n(t)$,

$$r(t) = s_i(t) + n(t)$$

- Since the filter is linear, the resulting output at the sampling instant $t = T$ may be expressed as

$$z(T) = a_i + n_0$$

- where a_i and n_0 are the signal component (deterministic) and noise component (random) of $r(t)$ respectively.

Digital Receivers

- The variance (or average power) of the output noise is σ^2.
- The signal-to-noise ratio (SNR) of the matched filter output at the sampling instant is

$$\left(\frac{S}{N}\right)_{t=T} = \frac{a_i^{\,2}}{\sigma^2}$$

- The requirement here is to find the optimum filter transfer function $H_0(f)$ that maximises $(S/N)_{t=T}$ in order to minimise the probability of making an incorrect decision, i.e. to minimise the probability of error.

Digital Receivers

- It can be shown that the expression for $(S/N)_{t=T}$ may be rewritten as

$$\left(\frac{S}{N}\right)_{t=T} \leq \frac{2E}{N_0}$$

- It is important to note that $(S/N)_{t=T}$ depends only on the signal energy E and the noise power spectral density $N_0/2$.

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- The maximum value of $(S/N)_{t=T}$ occurs when equality holds, i.e.

$$\left(\frac{S}{N}\right)_{t=T} = \frac{2E}{N_0}$$

- Correspondingly, the optimum value of $H(f)$, denoted $H_{opt}(f)$ occurs when

$$H_{opt}(f) = kS_i^*(f)e^{-j2\pi fT}$$

- where k is an arbitrary constant and * denotes the complex conjugate. $S_i(f)$ is the Fourier Transform of the signal waveform $s_i(t)$.

Digital Receivers

- In the time domain, the impulse response of the optimum filter $h_{opt}(t)$ is given by the inverse Fourier Transform of $H_{opt}(f)$, i.e.

$$
\begin{aligned}
h_{opt}(t) &= \Im^{-1}\left\{H(f)\right\} \\
&= \Im^{-1}\left\{kS_i^*(f)e^{-j2\pi fT}\right\} \\
&= ks_i(T-t)
\end{aligned}
$$

- The optimum filter will therefore have an impulse response that is a time-reversed and delayed version of the input signal $s_i(t)$.
- In other words, $h_{opt}(t)$ is “matched” to the input signal $s_i(t)$.

Digital Receivers

- A linear time-invariant filter defined in this way is termed a *matched filter*.
- The use of this matched filter will result the reception and detection of digital signals with the smallest probability of error.

Digital Receivers

- In summary, it has been shown that:
- A filter matched to an input signal $s_i(t)$ of duration T is characterised by an impulse response that is a time-reversed and delayed version of the input signal $s_i(t)$,

$$h_{opt}(t) = ks_i(T - t)$$

- In the frequency domain, the matched filter is characterised by a frequency response that is the complex conjugate of the Fourier transform of the input signal $s_i(t)$

$$H_{opt}(f) = kS_i^*(f)e^{-j2\pi fT}$$

Digital Receivers

- However, the most important result is that the maximum output signal-to-noise ratio of the matched filter depends only on the ratio of the signal energy E to the power spectral density of the white noise at the filter input.

$$\left(\frac{S}{N} \right)_{t=T} = \frac{2E}{N_0}$$

Digital Receivers

- In the time domain, a linear system is described in terms of its *impulse response* which is defined as the response of the system (with zero initial conditions) to a unit impulse $\delta(t)$ applied to the input of the system.
- If the system is *time invariant* then the shape of the impulse response is the same irrespective of when $\delta(t)$ was applied to the system.
- Suppose we have a linear time invariant system whose impulse response is given by $h(t)$*.* What will be the response of the system to an input signal $x(t)$?

Digital Receivers

- Unfortunately, the response is <u>not</u> something as simple as

$$y(t) = h(t)x(t)$$

- In fact, it is actually given by the *convolution integral* as follows

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= h(t) \otimes x(t)$$

Digital Receivers

- In the time domain the response is given by the *convolution* of the input signal and the system's impulse response. **Do not confuse with <u>correlation</u>.**
- However, if we *transform* the problem into the frequency domain, we have

$$Y(f) = H(X)X(f)$$

- where $X(f)$ and $Y(f)$ are the Fourier transforms of the input $x(t)$ and output $y(t)$ signals respectively, and $H(f)$ is the frequency response of the system which is given by the Fourier transform of the impulse response $h(t)$.

Digital Receivers

- In the time domain, the response is given by the *convolution* of the input signal and the system's impulse response.
- In the frequency domain, the response is given by the *product* of the input signal and the system's frequency response.
- By changing over between domains, we have greatly simplified the operation of calculating the response, i.e. the convolution integral has been replaced by a much simpler multiplication operation, i.e. we have *transformed* the problem.

Digital Receivers

- This is similar to the use of *logarithms* to transform the operations of multiplication, division, and exponentiation into (much simpler) addition, subtraction, and multiplication operations.

$$\log(AB) = \log A + \log B$$

$$\log(A/B) = \log A - \log B$$

$$\log(A^B) = B \log A$$

Digital Receivers

- *Question*: So, how we go between the time domain and frequency domain?

- *Answer*: Use the Fourier transform.

Digital Receivers

- Designing and realising filters is expensive since it requires a high degree of precision in terms of component values.
- Unfortunately, due to component tolerances in the manufacturing process, this is often difficult (or more specifically very costly) to achieve.
- Therefore, in practice the linear matched filter is more often implemented as a ***correlator***.
- There is an equivalence between the linear matched filter and the correlator that can be exploited to allow for a cheaper implementation.

Digital Receivers

- **Correlation** is essentially a pattern matching process where you are attempting to find if any *similarities* exist between two signals.
- Mathematically, correlation is the *integral of a product*, as follows:

$$z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, d\tau$$

- A large correlation value (positive or negative) implies a strong similarity between the two signals.
- A small correlation value represents a little similarity between the signals.

Digital Receivers

- The matched filter is often implemented as a correlator.
- The equivalence between the matched filter and correlator implementations may be shown as follows:
- The output of the linear matched filter is given by

$$z(t) = r(t) \otimes h(t)$$

- where $\otimes$ denotes <u>convolution</u> since we are dealing with the time domain (NOTE: convolution <u>not</u> correlation).

Digital Receivers

- The matched filter output $z(T)$ is given by

$$z(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau$$

- Substituting for $h(t)$ using (i.e. the impulse response of the matched filter)

$$h(t) = s(T - t)$$

- The matched filter output $z(T)$ now becomes

$$z(t) = \int_{0}^{T} r(\tau)s(T-(t-\tau))d\tau$$

$$= \int_{0}^{T} r(\tau)s(T-t+\tau)d\tau$$

Digital Receivers

- At the sampling instant $t = T$, the matched filter output $z(T)$ becomes

$$z(T) = \int_0^T r(\tau)s(\tau)d\tau$$

- The equivalent matched filter output (at the sampling instant) may be produced by correlating the received signal $r(t)$ with a “replica” of the original transmitted signal $s(t)$ for the duration of the signaling interval.

Digital Receivers

- This "replica" of the original signal is known as a *prototype* or *reference* signal that is generated locally at the receiver.

- **NOTE:** The correlator output and the matched filter output are <u>only</u> equal at the sampling instant $t = T$.

Digital Receivers

- In a matched filter receiver ,the received signal $r(t)$ is correlated with each of the transmitted signal prototypes $\{s_1(t),\ s_2(t),\ \ldots, s_M(t)\}$ using a bank of correlators.

- The prototype signal $s_i(t)$ whose corresponding correlator output $z_i(T)$ has the largest value corresponds to the original transmitted signal.

- In other words, the bank of correlators is used to determine <u>which</u> of prototype signals the received signal most closely matches.

Digital Receivers

- Before proceeding to investigate the error performance properties of the various signaling formats, it is necessary to first develop a framework for the analysis.
- Here, the *Signal Space Analysis* technique will be used.
- Signal space analysis permits a *geometric* representation of finite energy signals whereby signaling waveforms may be represented by vectors plotted in a signal space.