Detection in the Presence of AWGN Noise 2 – Continued

Lecture No. 15

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Overview

This lecture will look at the following:

- Probability of error
- A generic optimum receiver
- Matched filter versus correlator
- Application to baseband binary signalling, bipolar
Probability of Error

- An error occurs in the detection process in the receiver whenever, while signal $s_i$ has been sent, the received signal does not fall within the region $Z_i$ in the signal space.

- If $p_i$ is the probability for $s_i$ to be transmitted, the probability of error $P_e$ is the probability that the received signal $r$ is not in the region $Z_i$, when $s_i$ has been sent, averaged
over all \( i \), with the weight \( p_i \), giving:

\[
P_e = \sum_{i=1}^{M} p_i P(\text{r not in } Z_i | s_i \text{ sent})
\]

which can also be given as:

\[
1 - \sum_{i=1}^{M} p_i P(\text{r in } Z_i | s_i \text{ sent})
\]
which is given by:

\[ 1 - \sum_{i=1}^{M} p_i \int_{Z_i} f_r (r \mid s_i) \, dr \]

\( p_i \) can be replaced with \( 1/M \) if the signals \( s_i \) are equiprobable.
Generic Optimum Receiver
Matched Filter vs Correlator

- As already mentioned, the impulse response of the linear filter matching the signal $s(t)$ of duration $T$ can be written:

$$h(t) = s(T - t), \quad 0 \leq t \leq T$$

$$h(t) = 0, \quad \text{elsewhere}$$

The correlators in the above generic receivers can be replaced by filters matched to the signals $\Phi_i(t)$, i.e.:

$$h(t) = \Phi(T - t), \quad 0 \leq t \leq T$$

$$h(t) = 0, \quad \text{elsewhere}$$
For an input signal $x(t)$, the output of the filter matched to $\Phi(t)$ is identical at time $T$ only to the output of a correlator performing the integration of the product $x(t)\Phi_i(t)$ over the duration of the symbol $T$.

**Remember:**

$$z(t) = r(t) \otimes h(t) = \int_{0}^{t} r(\tau)h(t - \tau)d\tau$$
So we now have:

\[ z(t) = \int_{0}^{t} r(\tau) h(t - \tau) d\tau = \int_{0}^{t} r(\tau) \Phi_i [T - (t - \tau)] d\tau \]

This becomes, at \( t = T \):

\[ z(t) = \int_{0}^{t} r(\tau) \Phi_i [\tau] d\tau \]

which is the correlation of \( r(t) \) with \( \Phi_i(t) \)
Application to Baseband Binary Signalling – Bipolar

• Let us consider the case of bipolar baseband signalling

\[
\begin{align*}
    s_1(t) &= A \\
    s_2(t) &= -A
\end{align*}
\]

\[
0 \leq t \leq T
\]

Here we have \( M = 2 \) signals and we choose \( N \leq M = 1 \). From the representation of signals outlined early in
Lecture 13 we get:

\[ s_1(t) = s_{11} \Phi_1(t) \]
\[ s_2(t) = s_{21} \Phi_1(t) \]

From the Gram–Schmidt orthogonalisation we have:

\[ \Phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \]

\( E_1 \) is found from the equation for \( E_i \) given in Lecture 13, and is \( A^2T \), which combined with \( s_1(t) = A \) gives:

\[ \Phi_1(t) = \sqrt{\frac{1}{T}}, \quad \text{for } 0 \leq t \leq T \]
We therefore have:

\[ s_1(t) = A\sqrt{T}\Phi_1(t) = \sqrt{E_1}\Phi_1(t), \quad \text{and } s_{11} = \sqrt{E_1} \]

which gives:

\[ s_2(t) = -A\sqrt{T}\Phi_1(t) = -\sqrt{E_1}\Phi_1(t), \quad \text{and } s_{21} = -\sqrt{E_1} \]

- We therefore have the 1–dimensional signal space shown below with the distance between the 2 signals being \(2\sqrt{E_1} \).
Detection/Probability of Error:
Let us apply our generic results on the detection of the transmitted signal in AWGN.
• As \( N = 1 \), and both signals have the same energy, the criterion simplifies to:

\[ s_1 \text{ has been transmitted if } r_1 s_{k1} \text{ is a maximum when } k = i. \]  

So we have:

- \( s_1 \) has been sent if \( \sqrt{E_1} r_1 \) is greater than \( -\sqrt{E_1} r_1 \), i.e. \( r_1 > 0 \)

- \( s_2 \) has been sent if \( -\sqrt{E_1} r_1 \) is greater than \( \sqrt{E_1} r_1 \), i.e. if \( r_1 < 0 \)
The probability of error can be written:

\[ P_e = \sum_{i=1}^{M} p_i P(r_1 \text{ not in } Z_i \mid s_i \text{ sent}) \]

\[ = p_1 P(r_1 < 0 \mid s_1 \text{ sent}) + p_2 P(r_2 > 0 \mid s_2 \text{ sent}) \]
As $s_1$ and $s_2$ are equiprobable, $p_1 = p_2 = 1/2$.

$$P_e = \frac{1}{2} \int_{-\infty}^{0} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 - s_{11})^2}{2\sigma^2} \right) \, dr_1$$

$$+ \frac{1}{2} \int_{0}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 - s_{21})^2}{2\sigma^2} \right) \, dr_1$$
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\[ P_e = \frac{1}{2} \int_{\infty}^{0} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(-r_1 - s_{11})^2}{2\sigma^2} \right) (-dr_1) \]

\[ + \frac{1}{2} \int_{0}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 - s_{21})^2}{2\sigma^2} \right) dr_1 \]

\[ P_e = \frac{1}{2} \int_{0}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(-r_1 - s_{11})^2}{2\sigma^2} \right) dr_1 \]

\[ + \frac{1}{2} \int_{0}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 - s_{21})^2}{2\sigma^2} \right) dr_1 \]
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\[
P_e = \frac{1}{2} \int_0^\infty \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(-r_1 - \sqrt{E_1})^2}{2\sigma^2} \right) \, dr_1
\]

\[
+ \frac{1}{2} \int_0^\infty \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 + \sqrt{E_1})^2}{2\sigma^2} \right) \, dr_1
\]

\[
P_e = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(r_1 + \sqrt{E_1})^2}{2\sigma^2} \right) \, dr_1
\]

If we let:

\[u = \frac{(r_1 + \sqrt{E_1})}{\sigma}\]
we get the following expression for $P_e$:

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{\frac{\sqrt{E_1}}{\sigma}}^{\infty} \exp \left( -\frac{u^2}{2} \right) du = Q \left( \frac{\sqrt{E_1}}{\sigma} \right) = Q \left( \sqrt{\frac{2E_1}{N_0}} \right)$$

Which is exactly what we got in Lecture 12!!

- A bipolar baseband receiver has the block diagram shown below:
Choose $s_1$ if $r_1 > 0$
Choose $s_2$ if $r_1 < 0$

Bipolar baseband receiver
Conclusion

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