Matched Filter

Lecture No. 11

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Overview

This lecture will look at the following:

- Probability of error
- Error function
- Matched filter
Probability of Error

- There are 2 ways an error can occur:
  - $H_1$ is chosen while $s_2$ was sent
  - $H_2$ is chosen while $s_1$ was sent

- The total probability of error can be written:

$$P_B = p(H_2|s_1)p(s_1) + p(H_1|s_2)p(s_2)$$

As $s_1$ and $s_2$ are equiprobable ($p(s_1)=p(s_2)=1/2$), and because of the symmetry of the pdfs ($p(H_2|s_1)=p(H_1|s_2)$),
we have:

\[ P_B = p(H_1|s_2) = p(H_2|s_1) \]

\[ = p \left[ z(T) > \frac{(a_1 + a_2)}{2} | s_2 \right] \]

\[ = p \left[ z(T) < \frac{(a_1 + a_2)}{2} | s_1 \right] \]

We already know that the conditional probability density function of \( z \), given that \( s_2 \) was transmitted, is given by \( p(z|s_2) \). So we can find the probability that \( z > (a_1 + a_2)/2 \) (eventhough an \( s_2 \) (not an \( s_1 \) was sent) by integrating the pdf \( p(z|s_2) \) from \( (a_1 + a_2)/2 \) to infinity.
So the total probability is given by:

\[ P_B = \frac{1}{\sigma \sqrt{2\pi}} \int_{a_1+a_2}^{\infty} \exp \left( -\frac{1}{2} \left( \frac{z - a_2}{\sigma} \right)^2 \right) dz \]

We now let \( u = (z - a_2)/\sigma \) and get:

\[ P_B = \frac{1}{\sqrt{2\pi}} \int_{\frac{a_1-a_2}{2\sigma}}^{\infty} \exp \left( -\frac{u^2}{2} \right) dz \]

There is no closed form solution to this integral. The integral is evaluated using maths tables, numerical inte-
gration techniques or closed–form approximations. Here we use an error function and tables to evaluate the integral.

The probability of error $P_B$ is therefore given by:

$$P_B = Q \left( \frac{a_1 - a_2}{2\sigma} \right)$$

$Q$ is called the complementary or error function and must be tabulated.
The Error Function $Q$

- The $Q$–function is an integral of a Gaussian pdf defined as

$$Q(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp\left(-\frac{x^2}{2}\right) \, dx$$

- The $Q$–function has the following properties:
  - $Q(0) = 1/2$
  - $Q(-\infty) = 0$
  - $Q(+\infty) = 1$
- \( Q(-a) = 1 - Q(a) \)

- There are a number of similar integral functions used for error calculations. Common ones are:

\[
erf(a) = \frac{2}{\sqrt{\pi}} \int_{0}^{a} \exp(-x^2) \, dx, \quad a \geq 0
\]

\[
erfc(a) = \frac{2}{\sqrt{\pi}} \int_{a}^{\infty} \exp(-x^2) \, dx = 1 - erf\left(\frac{a}{\sqrt{2}}\right), \quad a \geq 0
\]
The $Q$ function is related to these functions by:

$$Q(a) = \frac{1}{2} \left[ 1 - erf \left( \frac{a}{\sqrt{\pi}} \right) \right] = \frac{1}{2} erf c \left( \frac{a}{\sqrt{2}} \right), \quad a \geq 0$$

$Q(a)$ or $Q(x)$ cannot be calculated directly and is available in tabular form.

It is possible to define a simple approximation which allows a quick evaluation, when $a > 3$:

$$Q(a) \approx \frac{1}{a\sqrt{2\pi}} \exp \left( -\frac{a^2}{2} \right), \quad a > 3$$

An example of some $Q$ function tables are given below:
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Complementary Error Function \( Q(x) = \int_0^x \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \, du \)

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<th>0.03</th>
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Matched Filter

- The matched filter is the linear filter providing the maximum signal-to-noise power ratio at its output for a given received symbol waveform.

- Let us again consider the description used for the output of our linear filter. The sampled output of the linear filter can be written:

\[ z(T) = a_i(T) + n(T) \]

where \( a_i(T) \) is the signal component, and \( n(T) \), the noise
component.

- The ratio of the instantaneous signal power to the average noise power, at time $T$ at the output of the receiver is:

$$\frac{S}{N}_T = \frac{a_i^2}{n^2}$$

- The signal at the output of the filter can be described using the inverse Fourier transform of its spectrum:

$$a(t) = \int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi ft} df$$
where $S(f)$ is the Fourier transform of the signal at the input of the filter, and $H(f)$ is the filter transfer function.

- The noise power at the output of the filter can be written:

$$n^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

**Remember:** The power spectral density of white noise is $N_0/2$

- We can rewrite the signal–to–noise ratio at the output of
the receiver at time $T$ as:

$$
\left( \frac{S}{N} \right)_T = \frac{\left| \int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi f T} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}
$$

**Note:** We can use the Schwarz’s inequality:

$$
\left| \int_{-\infty}^{\infty} f(x) g(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} |g(x)|^2 dx
$$

which holds for $f(x) = k g^*(x)$, with $k$ an arbitrary con-
stant and * indicating complex conjugate

Using Schwarz’s inequality gives:

\[
\left( \frac{S}{N} \right)_T = \frac{\left| \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{\int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |S(f)|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}
\]
This simplifies to:

$$\left( \frac{S}{N} \right)_T = \frac{2}{N_0} \int_{-\infty}^{\infty} \left| S(f) \right|^2 df$$

We can identify the energy of the input signal as:

$$E = \int_{-\infty}^{\infty} \left| S(f) \right|^2 df$$

**Remember:** The equation for energy spectral density given in Lecture 5
So we can write the previous inequality as:

$$\max \left( \frac{S}{N} \right)_T = \frac{2E}{N_0}$$

The maximum signal to noise ratio depends on the energy of the input signal, and the power spectral density of the noise.

According to Schwarz’s theorem this equality holds for

$$H(f) = kS^*(f)e^{-j2\pi ft}$$

Taking the inverse Fourier transform, we get the time
domain response:

\[ h(t) = \begin{cases} 
    ks(T - t), & \text{for } 0 < t < T; \\
    0, & \text{elsewhere.}
\end{cases} \]

- **Matched Filter Summary:**
  A filter that is matched to a signal \( s(t) \) of duration \( T \), has an impulse response that is a time–reversed and delayed version of the input \( s(t) \).

The impulse response of the linear filter matching the
signal \( s(t) \) of duration \( T \) can be written:

\[
h(t) = \begin{cases} 
  s(T - t), & \text{for } 0 < t < T; \\
  0, & \text{elsewhere.}
\end{cases}
\]

The peak pulse signal–to–noise ratio depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input. The peak pulse signal–to–noise ratio is \((S/N)_{\text{max}} = 2E/N_0\), where \(N_0/2\) is the power spectral density of the input noise.
Conclusion

This lecture has looked at the following:

- Probability of error
- Error function
- Matched filter